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Analytical approach for the Floquet theory of delay differential equations

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We present an analytical approach to deal with nonlinear delay differential equations close to instabilities of time periodic reference states. To this end we start with approximately determining such reference states by extending the Poincaré-Lindstedt and the Shohat expansions, which were originally developed for ordinary differential equations. Then we systematically elaborate a linear stability analysis around a time periodic reference state. This allows us to approximately calculate the Floquet eigenvalues and their corresponding eigensolutions by using matrix valued continued fractions. [S1063-651X(99)14005-4]

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I. INTRODUCTION

Over the last two decades considerable new interest in the theory of delay differential equations has led to various remarkable results [1–4]. The reason is that the solution space for delay differential equations has to be considered as infinite dimensional, although only a finite number of dynamical variables is involved [5]. As a consequence, nonlinear delay differential equations reveal a broad class of instabilities leading from oscillatory to chaotic behavior. Apart from the period doubling route to chaos, quasiperiodic states, intermittency, and locking behavior have also been observed in detailed numerical studies [4]. In particular, in the chaotic domain it has been suggested that the envelope to the Kaplan-Yorke dimension of a delay induced chaotic attractor is proportional to the time delay [4,6–8]. This fact offers the possibility of generating high dimensional chaotic attractors by simply increasing the time delay.

Delay differential equations have been successfully applied to model numerous nonlinear systems where dynamical instabilities are induced by the finite propagation time of signals in feedback loops. For instance, experiments on optical devices, acousto-optic and electro-optic bistable devices [9–11] have confirmed both the theoretical and numerical predictions. But delay induced instabilities also play an important role in other disciplines, such as population dynamics [1], radio engineering sciences [12], economics [13], and biology [14]. In addition, it has been noted in medical sciences that there exists a remarkable variety of clinically relevant dynamical phenomena under physiological and pathological conditions. For example, oscillations or chaotic behavior can spontaneously occur or disappear as a function of external or internal time delays, as has been demonstrated by the Mackey-Glass model of blood circulation [15], the Cheyne-Stokes respiration [16], and the forearm tracking with visual delayed feedback [17].

The interesting properties of nonlinear delay differential equations have been mainly investigated in numerical studies. Therefore it becomes desirable to substantiate these results—at least in comparably simple situations—by analytical methods. An interesting result in this direction has been recently obtained in [3] by rigorously analyzing the instability of a time independent reference state. The application of the theory to a delay induced Hopf bifurcation has been confirmed by numerical as well as experimental studies. We note that a different method, which is based upon a multiple scaling analysis, has been recently demonstrated in [18]. Here, however, it is our aim to generalize the approach of [3] by starting from a time periodic reference state and by analytically investigating its stability.

Our paper is organized as follows. In Sec. II we introduce two methods for approximately determining a time periodic reference state. Section III then develops its linear stability analysis. The resulting Floquet theory leads to a homogeneous vector valued recurrence relation determining the Floquet eigenvalues and its corresponding eigensolutions. In Sec. IV we offer two solution methods for this recurrence relation that are based on matrix valued continued fractions. Section V completes the Floquet theory by studying the adjoint problem. Eventually Sec. VI is devoted to a short summary and several conclusions in view of possible future work. For a numerical derivation of the Floquet exponents and the corresponding eigenvectors for this case we refer to [19,20].

II. DETERMINATION OF THE TIME PERIODIC REFERENCE STATE

We assume that the dynamical behavior of the system under consideration can be characterized by a state vector $\vec{q}(t)$ in an n dimensional state space Γ and that the underlying equation of motion is an autonomous delay differential equation of the general form

$$\frac{d}{dt}\vec{q}(t)=\vec{N}(\vec{q}(t),\vec{q}(t-\tau),\{\sigma_i\}). \tag{1}$$

Here $\vec{N}$ denotes a nonlinear vector field that depends on the state vector $\vec{q}$ at the times t and $t-\tau$, respectively, with τ representing the time delay. The set $\{\sigma_i\}$ describes the control parameters that measure external influences on the system. We assume that these control parameters are kept fixed so that we can omit them in our notation.

The treatment of the nonlinear problem (1) close to an instability strongly depends on the chosen reference state. A theory for an instability of a time independent reference state has been recently developed in [3]. Here it is our aim to generalize this method to situations where we start from a time periodic reference state. As such states cannot be expressed by closed analytical forms, it becomes necessary to describe them by using proper approximation schemes. In this section we extend methods that have been developed in the realm of ordinary differential equations towards delay differential equations. The Poincaré-Lindstedt approximation allows us to determine the time periodic reference state for small values of a parameter, whereas its improvement, the Shohat method turns out to possess a wider range of applicabilities [21,22].

A. The Poincaré-Lindstedt expansion

A perturbative approximation method, such as the Poincaré-Lindstedt expansion, relies on the existence of a suitable smallness parameter μ. Dealing with the nonlinear differential equation (1), we have to distinguish in general two different origins for such a smallness parameter μ. On the one hand, the smallness parameter μ can be generated by a delay induced instability. Then it measures the relative deviation of the time delay τ from the critical value τ_c above which the delay induced time periodic reference state exists. This case occurs, for instance, in the electronic phase locked loop with time delay [3] where the underlying model equation reveals a Hopf bifurcation at some τ_c. Considering the corresponding normal form [23]

$$\frac{dZ}{dt}=\sigma Z-g|Z|^2Z, \tag{2}$$

we may choose $\mu=\sqrt{(\tau-\tau_c)/\tau_c}$. On the other hand, the smallness parameter μ can also coincide with one of the given control parameters of the system. An example is provided by a harmonic oscillator with frequency ω_0, which is driven by a nonlinear time delayed perturbation:

$$\begin{aligned}&q''(t,\mu)+\omega_0^2q(t,\mu)\\&\quad=\mu f(q(t,\mu),q'(t,\mu),q(t-\tau,\mu),q'(t-\tau,\mu)).\end{aligned} \tag{3}$$

Here $q(t,\mu)$ denotes a scalar variable, the prime abbreviates the derivative with respect to the time t, and f represents a nonlinear function of its arguments.

For the sake of simplicity we now discuss the Poincaré-Lindstedt expansion, not for the general delay differential equation (1), but only for the model equation (3). We start with the situation of a vanishing smallness parameter μ where the solution of (3) is a periodic reference state

$$q(t,0)=q(t+T_0,0), \tag{4}$$

with $T_0=2\pi/\omega_0$ denoting the period of the unperturbed oscillator. Switching on the smallness parameter μ, this state will be transformed to a new periodic state, which can be described by

$$q(t,\mu)=q\left(t+\frac{2\pi}{\omega(\mu)},\mu\right). \tag{5}$$

In the following it becomes useful to explicitly take into account the frequency shift from ω_0 to $\omega(\mu)$ by rescaling the time t according to

$$\xi(t)=\omega(\mu)t. \tag{6}$$

Introducing the new variable

$$x(\xi,\mu)=q\left(\frac{\xi}{\omega(\mu)},\mu\right), \tag{7}$$

which is 2π periodic in ξ,

$$x(\xi,\mu)=x(\xi+2\pi,\mu), \tag{8}$$

we can rewrite the equation of motion (3) as

$$\begin{aligned}&\omega(\mu)^2\ddot{x}(\xi,\mu)+\omega_0^2x(\xi,\mu)\\&\quad=\mu f(x(\xi,\mu),\dot{x}(\xi,\mu),x(\xi-\omega(\mu)\tau,\mu),\dot{x}(\xi-\omega(\mu)\tau,\mu)).\end{aligned} \tag{9}$$

The dot indicates the derivative with respect to the dimensionless new time variable ξ.

As already mentioned, we assume that μ represents a small quantity so that we can expand the frequency $\omega(\mu)$ and the periodic orbit $x(\xi,\mu)$ in powers of μ according to

$$x(\xi,\mu)=x_0(\xi)+\mu x_1(\xi)+\mu^2x_2(\xi)+\cdots, \tag{10}$$

$$\omega(\mu)=\omega_0+\mu\omega_1+\mu^2\omega_2+\cdots. \tag{11}$$

In addition to the similar procedure for ordinary differential equations [21,22], we have also to consider a corresponding expansion of the time delayed terms in Eq. (9). This is achieved by

$$\begin{aligned}x(\xi-\omega(\mu)\tau,\mu)=&\,x_0(\xi-\omega_0\tau)+\mu(x_1(\xi-\omega_0\tau)\\&-\omega_1\tau\dot{x}_0(\xi-\omega_0\tau))+\cdots,\end{aligned} \tag{12}$$

$$\begin{aligned}\dot{x}(\xi-\omega(\mu)\tau,\mu)=&\,\dot{x}_0(\xi-\omega_0\tau)+\mu(\dot{x}_1(\xi-\omega_0\tau)\\&-\omega_1\tau\ddot{x}_0(\xi-\omega_0\tau))+\cdots.\end{aligned} \tag{13}$$

If we apply these expansions to the equation of motion (9) and combine terms of the same power of μ, we obtain in each order a system of inhomogeneous linear ordinary differential equations of second order:

$$\ddot{x}_0(\xi)+x_0(\xi)=0, \tag{14}$$

$$\ddot{x}_1(\xi)+x_1(\xi)$$
$$=-2\frac{\omega_1}{\omega_0}\ddot{x}_0(\xi)$$
$$+\frac{1}{\omega_0^2}f(x_0(\xi),\dot{x}_0(\xi),x_0(\xi-\omega_0\tau),\dot{x}_0(\xi-\omega_0\tau)), \quad (15)$$
$$\vdots$$
$$\ddot{x}_n(\xi)+x_n(\xi)=I_n(\xi). \quad (16)$$

The inhomogeneity $I_n(\xi)$, which appears in the nth order (16), is purely determined by the lower order terms $x_m(\xi)$, $0\leqslant m<n$. We have to guarantee that our periodicity condition (8) is fulfilled in each order of the perturbation theory. However, if the Fourier expansion of the inhomogeneity $I_n(\xi)$ includes multiples of the first harmonic terms that are proportional to $\sin(\xi)$ or $\cos(\xi)$, the solution $x_n(\xi)$ of Eq. (16) contains aperiodic secular terms of the form $\xi\sin(\xi)$ or $\xi\cos(\xi)$, respectively. We can avoid these aperiodic solutions by demanding

$$\int_0^{2\pi}I_n(\xi)\sin(\xi)d\xi=0, \quad \int_0^{2\pi}I_n(\xi)\cos(\xi)d\xi=0. \quad (17)$$

In order to fulfill these two conditions we need two independent parameters. Here we choose the constant ω_n as the first parameter, whereas the second one can be chosen by imposing suitable initial conditions for $x_{n-1}(\xi)$; for example,

$$x_{n-1}(0)=A_{n-1}, \quad \dot{x}_{n-1}(0)=0. \quad (18)$$

In this way we obtain a systematic approximation scheme to determine our time periodic reference state order by order for small values of the parameter μ.

B. The Shohat expansion

In a situation where the parameter μ is not a small quantity, the Poincaré-Lindstedt expansion for the calculation of the time periodic reference state has to be modified. This can be achieved by introducing a new smallness parameter $\rho(\mu)$ by the prescription

$$\rho(\mu)=\frac{\mu}{1+\mu}, \quad (19)$$

which maps the interval $[0,\infty)$ of μ onto the interval $[0,1)$ of ρ. The resulting method of the Shohat expansion can be described as follows. The equation of motion (9) is multiplied by μ^2. In doing so we become able to expand the periodic reference state $x(\xi,\mu)$ as well as the product $\mu\omega(\mu)$ with respect to ρ and obtain

$$x(\xi,\mu)=X_0(\xi)+\rho(\mu)X_1(\xi)+\rho(\mu)^2X_2(\xi)+\cdots, \quad (20)$$

$$\mu\omega(\mu)=\rho(\mu)\Omega_0+\rho(\mu)^2\Omega_1+\rho(\mu)^3\Omega_2+\cdots. \quad (21)$$

In order to guarantee that the frequency $\omega(\mu)$ approaches the frequency ω_0 of the unperturbed harmonic oscillator in the limit $\mu\to 0$ we have to choose $\Omega_0=\omega_0$. From Eqs. (20) and (21) and the inversion of the relation (19),

$$\mu=\frac{\rho(\mu)}{1-\rho(\mu)}, \quad (22)$$

we deduce the expansions

$$\omega(\mu)=\Omega_0+\rho(\mu)(\Omega_1-\Omega_0)+\rho(\mu)^2(\Omega_2-\Omega_1)+\cdots, \quad (23)$$

$$x(\xi-\omega(\mu)\tau,\mu)=X_0(\xi-\Omega_0\tau)+\rho(\mu)[X_1(\xi-\Omega_0\tau)-(\Omega_1-\Omega_0)\tau\dot{X}_0(\xi-\Omega_0\tau)]+\cdots. \quad (24)$$

The further application of the Shohat method is completely analogous to the Poincaré-Lindstedt approximation scheme. It has been conjectured, without proof [21], that the method works for arbitrary parameter values $\mu\geqslant 0$. We thus note that, although to our knowledge no known counterexample for this conjecture exists, the validity of this expansion has to be confirmed for each case individually.

III. STABILITY OF THE TIME PERIODIC REFERENCE STATE

We now generalize the linear stability analysis of a time independent reference state developed in [3] to the case of a time periodic reference state. To that end we return to the general form of the delay differential equation (1) and rescale the time according to Eqs. (6) and (7):

$$\frac{d}{d\xi}\vec{q}(\xi)=\frac{1}{\omega}\vec{N}(\vec{q}(\xi),\vec{q}(\xi-\omega\tau),\{\sigma_i\}). \quad (25)$$

Here $\omega=\omega(\mu)$ abbreviates the frequency of the time periodic reference state, henceforth denoted by $\vec{q}^{\,0}(\xi)=\vec{q}^{\,0}(\xi,\mu)$.

Following the original notion of Krasovskii and Hale [1,5], as well as its detailed elaboration in [3], we generalize the n dimensional state space Γ to an infinite dimensional state space $\mathcal{C}$. This allows us to embed the given delay differential equation (25) in the context of functional differential equations. It turns out that this reformulation represents an adequate framework for a linear stability analysis around a time periodic reference state. The resulting Floquet theory leads to a homogeneous vector valued recurrence relation that determines the Floquet eigenvalues as well as the corresponding Floquet eigensolutions.

A. Formulation of the problem in the extended state space

It appears that solutions of the delay differential equation (25) for times $\xi\geqslant 0$ depend on initial values of the state vector $\vec{q}(\xi)$ in the entire interval $[-\omega\tau,0]$. Therefore we have to complete Eq. (25) with the initial condition

$$\vec{q}(\theta)=\vec{g}(\theta), \quad -\omega\tau\leqslant\theta\leqslant 0, \quad (26)$$

where $\vec{g}$ is a given continuous vector valued function in a suitable function space $\mathcal{C}$. The initial value problem [Eqs.

(25) and (26)] then maps the function $\vec{g}$ onto a trajectory in the n dimensional state space Γ. Therefore the problem arises that different initial vector valued functions $\vec{g}$ may yield crossings of the corresponding trajectories in Γ. This means that the pointwise uniqueness of solutions cannot be assured when we restrict our considerations to the state space Γ.

In order to solve this problem one may introduce the extension of the finite dimensional state space Γ to an infinite dimensional function space $\mathcal{C}$ where the initial vector valued function $\vec{g}$ is defined. According to Krasovskii and Hale [1,5] this is achieved by regarding the trajectory $\vec{q}(\xi)$ in the original state space Γ during the time interval $[\xi-\omega\tau,\xi]$ as a single point $\vec{q}_\xi$ in the extended space $\mathcal{C}$:

$$\vec{q}_\xi(\theta)=\vec{q}(\xi+\theta), \quad -\omega\tau\leqslant\theta\leqslant 0. \tag{27}$$

The dynamics of the delay system can then also be described in the extended state space $\mathcal{C}$ by introducing the nonlinear solution operator $\mathcal{T}(\xi)$:

$$\vec{q}_\xi(\theta)=(\mathcal{T}(\xi)\vec{g})(\theta), \quad -\omega\tau\leqslant\theta\leqslant 0. \tag{28}$$

Its uniqueness is expressed by the fact that the operator $\mathcal{T}(\xi)$ has the properties of a semigroup; that is,

$$\mathcal{T}(\xi+\eta)=\mathcal{T}(\xi)\mathcal{T}(\eta), \quad \xi,\eta\geqslant 0, \quad \mathcal{T}(0)=\mathcal{I}, \tag{29}$$

where $\mathcal{I}$ denotes the identity operator. We now have to reformulate the original initial value problem [Eqs. (25) and (26)] in the extended space $\mathcal{C}$. To this end we formally differentiate Eq. (28) with respect to the time ξ,

$$\frac{d}{d\xi}\vec{q}_\xi(\theta)=(\mathcal{A}\vec{q}_\xi)(\theta), \quad -\omega\tau\leqslant\theta\leqslant 0. \tag{30}$$

Here $\mathcal{A}$ denotes the infinitesimal generator that corresponds to the solution operator $\mathcal{T}(\xi)$:

$$(\mathcal{A}\vec{q}_\xi)(\theta)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\{[\mathcal{T}(\epsilon)\vec{q}_\xi](\theta)-\vec{q}_\xi(\theta)\}. \tag{31}$$

By evaluating this limit separately for the interval $-\omega\tau\leqslant\theta<0$ and for the point $\theta=0$ we obtain the explicit expression [3]

$$(\mathcal{A}\vec{q}_\xi)(\theta)=\begin{cases}\dfrac{d}{d\theta}\vec{q}_\xi(\theta), & -\omega\tau\leqslant\theta<0,\\[2mm] \mathcal{N}[\vec{q}_\xi(\cdot)], & \theta=0.\end{cases} \tag{32}$$

The nonlinear functional $\mathcal{N}$ is constructed as follows. We assume that the original vector field $\vec{N}$ in Eq. (25) can be expanded into powers of its arguments $\vec{q}(\xi)$ and $\vec{q}(\xi-\omega\tau)$. A typical term of second order in this expansion has, for instance, the form

$$N^{(2)}_{ijk}q_j(\xi)q_k(\xi-\omega\tau), \tag{33}$$

where the explicit components of the respective vectors have been introduced and summation is understood over dummy indices. The representation of $\vec{q}(\xi)$ and $\vec{q}(\xi-\omega\tau)$ can be given in terms of the extended state space $\mathcal{C}$ by taking into account the relation (27):

$$\vec{q}(\xi)=\int_{-\omega\tau}^{0}d\theta\,\delta(\theta)\vec{q}_\xi(\theta),$$

$$\vec{q}(\xi-\omega\tau)=\int_{-\omega\tau}^{0}d\theta\,\delta(\theta+\omega\tau)\vec{q}_\xi(\theta). \tag{34}$$

If we apply this procedure to every term in the series expansion and collect terms of the same order in the extended state vector $\vec{q}_\xi$, the nonlinear vector field $\vec{N}$ becomes a vector valued functional $\mathcal{N}$ with the components

$$\begin{aligned}\mathcal{N}_i[\vec{q}_\xi(\cdot)]=\sum_{k=1}^{\infty}\int_{-\omega\tau}^{0}d\theta_1\cdots\\ \times\int_{-\omega\tau}^{0}d\theta_k\frac{1}{\omega}\Omega^{(k)}_{i,j_1\ldots j_k}(\theta_1,\ldots,\theta_k)\\ \times q_{\xi,j_1}(\theta_1)\cdots q_{\xi,j_k}(\theta_k),\end{aligned} \tag{35}$$

where the $\Omega^{(k)}_{i,j_1\ldots j_k}(\theta_1,\ldots,\theta_k)$ represent matrix valued densities. Thus we have reached our first goal, namely, to derive a nonlinear functional differential equation for the problem formulated in Eqs. (25) and (26).

B. The linearized equation of motion

According to the prescription (27) the time periodic reference state $\vec{q}^{\,0}(\xi)$ in the state space Γ transforms into $\vec{q}^{\,0}_\xi(\theta)$ in the extended state space $\mathcal{C}$. In order to test its linear stability we insert the ansatz

$$\vec{q}_\xi(\theta)=\vec{q}^{\,0}_\xi(\theta)+\tilde{\vec{q}}_\xi(\theta) \tag{36}$$

into Eqs. (30) and (32). Dropping the tilde we obtain in the linear approximation for the infinitesimal deviation $\vec{q}_\xi(\theta)$,

$$\frac{d}{d\xi}\vec{q}_\xi(\theta)=(\mathcal{A}_L\vec{q}_\xi)(\theta), \tag{37}$$

where the linear infinitesimal generator $\mathcal{A}_L$ becomes explicitly time dependent:

$$(\mathcal{A}_L\vec{q}_\xi)(\theta)=\begin{cases}\dfrac{d}{d\theta}\vec{q}_\xi(\theta), & -\omega\tau\leqslant\theta<0,\\[2mm] \displaystyle\int_{-\omega\tau}^{0}d\theta'\,\mathbf{\Omega}_\xi(\theta')\vec{q}_\xi(\theta'), & \theta=0.\end{cases} \tag{38}$$

The matrix valued density $\mathbf{\Omega}_\xi(\theta)$ can be written as a functional derivative of $\mathcal{N}$ evaluated at the time periodic reference state $\vec{q}^{\,0}_\xi$:

$$\mathbf{\Omega}_\xi(\theta)=\left[\frac{\delta\mathcal{N}[\vec{q}_\xi(\cdot)]}{\delta\vec{q}_\xi(\theta)}\right]_{\vec{q}_\xi=\vec{q}^{\,0}_\xi}. \tag{39}$$

C. Transformation of the linear problem

Due to the fact that the reference state $\vec{q}^{\,0}_{\xi}(\theta)$ is 2π periodic with respect to ξ, the matrix valued density $\mathbf{\Omega}_{\xi}(\theta)$ in Eq. (39) is time dependent with the same period. We thus perform a Fourier expansion of the matrix $\mathbf{\Omega}_{\xi}(\theta)$:

$$\mathbf{\Omega}_{\xi}(\theta)=\sum_{k=-\infty}^{\infty}\mathbf{\Omega}_k(\theta)e^{ik\xi}. \tag{40}$$

In close analogy to the Floquet theorem for ordinary differential equations [24] we try to solve Eqs. (37)–(40) by the ansatz

$$\vec{q}_{\xi}(\theta)=e^{\lambda\xi}\vec{\phi}^{\lambda}_{\xi}(\theta). \tag{41}$$

Here λ denotes the Floquet eigenvalue and $\vec{\phi}^{\lambda}_{\xi}(\theta)=\vec{\phi}^{\lambda}_{\xi+2\pi}(\theta)$ is a 2π periodic Floquet eigensolution for which we also perform a Fourier expansion:

$$\vec{\phi}^{\lambda}_{\xi}(\theta)=\sum_{n=-\infty}^{\infty}\vec{\phi}^{\lambda}_n(\theta)e^{in\xi}. \tag{42}$$

In order to determine the Fourier components $\vec{\phi}^{\lambda}_n(\theta)$ we insert the hypothesis (41), (42) into the linearized equation of motion (37)–(40). We now have to consider separately the interval $-\omega\tau\leqslant\theta<0$ and the point $\theta=0$. In the interval $-\omega\tau\leqslant\theta<0$ we conclude that the Fourier component $\vec{\phi}^{\lambda}_n(\theta)$ has the form

$$\vec{\phi}^{\lambda}_n(\theta)=\vec{\phi}^{\lambda}_n e^{(\lambda+in)\theta}. \tag{43}$$

For the case $\theta=0$ we find

$$\sum_{n=-\infty}^{\infty}\vec{\phi}^{\lambda}_n(\lambda+in)e^{(\lambda+in)\xi}=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\int_{-\omega\tau}^{0}d\theta\,\mathbf{\Omega}_k(\theta)e^{(\lambda+in)\theta}e^{(i(k+n)+\lambda)\xi}\vec{\phi}^{\lambda}_n. \tag{44}$$

We now introduce the matrix valued quantity

$$\mathbf{L}_{k,n}=\int_{-\omega\tau}^{0}d\theta\,\mathbf{\Omega}_k(\theta)e^{(\lambda+in)\theta} \tag{45}$$

and a new index $\tilde{n}(n)=n+k$. Comparing the contributions of the various Fourier components in Eq. (44) and dropping the tilde, we obtain a homogeneous vector valued recurrence relation for the Fourier components $\vec{\phi}^{\lambda}_n$:

$$0=\sum_{k=-\infty}^{\infty}[\mathbf{L}_{k,n-k}-\delta_{k,0}(\lambda+in)\mathbf{I}]\vec{\phi}^{\lambda}_{n-k}. \tag{46}$$

Thus we are left with the problem of constructing an approximate solution to Eq. (46) that leads to both the Floquet eigenvalues λ and the corresponding Floquet eigensolutions.

D. Remark

In the Floquet theory of ordinary differential equations [24] it is shown that the derivative of the time periodic reference state represents a Floquet eigensolution where the real part of the corresponding Floquet eigenvalue vanishes. This statement remains valid for delay differential equations as can be seen as follows. As the time periodic reference state $\vec{q}^{\,0}_{\xi}(\theta)$ satisfies the nonlinear equation of motion (30), (32), a differentiation with respect to the time ξ leads to

$$\frac{d}{d\xi}\frac{d\vec{q}^{\,0}_{\xi}(\theta)}{d\xi}=\begin{cases}\dfrac{d}{d\theta}\dfrac{d\vec{q}^{\,0}_{\xi}(\theta)}{d\xi}, & -\omega\tau\leqslant\theta<0,\\[2ex] \displaystyle\int_{-\omega\tau}^{0}d\theta'\left[\frac{\delta\mathcal{N}[\vec{q}_{\xi}(\cdot)]}{\delta\vec{q}_{\xi}(\theta')}\right]_{\vec{q}_{\xi}=\vec{q}^{\,0}_{\xi}}\frac{d\vec{q}^{\,0}_{\xi}(\theta')}{d\xi}, & \theta=0.\end{cases} \tag{47}$$

A comparison with Eqs. (37)–(39) reveals that the derivative of the time periodic reference state $\vec{q}^{\,0}_{\xi}(\theta)$ indeed fulfills the linear problem. Due to Eqs. (41) and (42) it therefore possesses the general form

$$\frac{d\vec{q}^{\,0}_{\xi}(\theta)}{d\xi}=e^{\lambda\xi}\sum_{n=-\infty}^{\infty}\vec{\phi}^{\lambda}_n(\theta)e^{in\xi}. \tag{48}$$

From the 2π periodicity of $\vec{q}^{\,0}_{\xi}(\theta)$ and its derivative (48), we conclude that the real part of its Floquet eigenvalue λ has to vanish.

IV. MATRIX VALUED CONTINUED FRACTIONS

We consider two methods that enable us to approximately solve Eq. (46) for the Floquet eigenvalues λ and for the Fourier components $\vec{\phi}^{\lambda}_n$ of the Floquet eigensolutions. In the first part we formulate a new method based on n diagonal continued fractions. In the second part we show that this solution method is equivalent to a formulation with tridiagonal continued fractions introduced by Risken [25]. It turns out, however, that the first method is much simpler to handle, as the necessary inversion of matrices can be performed in a low dimensional space. Furthermore, the criterion for truncating higher order terms in the smallness parameter μ can be formulated more precisely in the framework of the first method.

A. Pentadiagonal recurrence relations

We apply the method of matrix valued continued fractions in order to solve the vector valued recurrence relation (46) approximately. In order to avoid overloading the notation, we restrict ourselves for the time being to the pentadiagonal case where the summation in Eq. (46) is performed for $-2\leqslant k\leqslant 2$:

$$0=\mathbf{L}_{-2,n+2}\vec{\phi}^{\lambda}_{n+2}+\mathbf{L}_{-1,n+1}\vec{\phi}^{\lambda}_{n+1}+[\mathbf{L}_{0,n}-(\lambda+in)\mathbf{I}]\vec{\phi}^{\lambda}_n+\mathbf{L}_{1,n-1}\vec{\phi}^{\lambda}_{n-1}+\mathbf{L}_{2,n-2}\vec{\phi}^{\lambda}_{n-2}. \tag{49}$$

We start by defining a set of ladder operators $\mathbf{S}_n^m$ for $m = \pm 1, \pm 2$ that relate neighboring Fourier components via

$$\vec{\phi}_{n+m}^{\lambda} = \mathbf{S}_n^m \vec{\phi}_n^{\lambda}. \tag{50}$$

This definition implies the following useful relations between different ladder operators:

$$\mathbf{S}_n^{-1} = [\mathbf{S}_{n-1}^{+1}]^{-1}, \quad \mathbf{S}_{n+1}^{+1}\mathbf{S}_n^{+1} = \mathbf{S}_n^{+2}. \tag{51}$$

Applying the definition (50) of the ladder operators, the pentadiagonal recurrence relation (49) can be rewritten as

$$0 = \{\mathbf{L}_{-2,n+2}\mathbf{S}_n^{+2} + \mathbf{L}_{-1,n+1}\mathbf{S}_n^{+1} + [\mathbf{L}_{0,n} - (\lambda + in)\mathbf{I}] + \mathbf{L}_{1,n-1}\mathbf{S}_n^{-1} + \mathbf{L}_{2,n-2}\mathbf{S}_n^{-2}\}\vec{\phi}_n^{\lambda}. \tag{52}$$

We now express the ladder operators $\mathbf{S}_n^m (m = \pm 1, \pm 2)$ in terms of the matrices $\mathbf{L}_{k,n}$ as well as the operators $\mathbf{S}_{n\pm1}^m, \mathbf{S}_{n\pm2}^m$. In order to evaluate this dependence, we isolate the term $\vec{\phi}_{n+1}^{\lambda} = \mathbf{S}_n^{+1}\vec{\phi}_n^{\lambda}$ in Eq. (52). Then the equation assumes the form

$$\mathbf{L}_{-1,n+1}\vec{\phi}_{n+1}^{\lambda} = -\{[\mathbf{L}_{0,n} - (\lambda + in)\mathbf{I}] + \mathbf{L}_{-2,n+2}\mathbf{S}_n^{+2} + \mathbf{L}_{1,n-1}\mathbf{S}_n^{-1} + \mathbf{L}_{2,n-2}\mathbf{S}_n^{-2}\}\vec{\phi}_n^{\lambda}. \tag{53}$$

Shifting the index from n to $n-1$ and applying the definition $\vec{\phi}_{n-1}^{\lambda} = \mathbf{S}_n^{-1}\vec{\phi}_n^{\lambda}$, we obtain from the validity for all $\vec{\phi}_n^{\lambda}$ the operator relation

$$\mathbf{S}_n^{-1} = -(\{\mathbf{L}_{0,n-1} - [\lambda + i(n-1)]\mathbf{I}\} + \mathbf{L}_{-2,n+1}\mathbf{S}_{n-1}^{+2} + \mathbf{L}_{1,n-2}\mathbf{S}_{n-1}^{-1} + \mathbf{L}_{2,n-3}\mathbf{S}_{n-1}^{-2})^{-1}\mathbf{L}_{-1,n}. \tag{54}$$

Similarly we construct the operator relations

$$\mathbf{S}_n^{+1} = -(\{\mathbf{L}_{0,n+1} - [\lambda + i(n+1)]\mathbf{I}\} + \mathbf{L}_{-2,n+3}\mathbf{S}_{n+1}^{+2} + \mathbf{L}_{-1,n+2}\mathbf{S}_{n+1}^{+1} + \mathbf{L}_{2,n-1}\mathbf{S}_{n+1}^{-2})^{-1}\mathbf{L}_{1,n}, \tag{55}$$

$$\mathbf{S}_n^{-2} = -(\{\mathbf{L}_{0,n-2} - [\lambda + i(n-2)]\mathbf{I}\} + \mathbf{L}_{-1,n-1}\mathbf{S}_{n-2}^{+1} + \mathbf{L}_{1,n-3}\mathbf{S}_{n-2}^{-1} + \mathbf{L}_{2,n-4}\mathbf{S}_{n-2}^{-2})^{-1}\mathbf{L}_{-2,n}, \tag{56}$$

$$\mathbf{S}_n^{+2} = -(\{\mathbf{L}_{0,n+2} - [\lambda + i(n+2)]\mathbf{I}\} + \mathbf{L}_{-1,n+3}\mathbf{S}_{n+2}^{+1} + \mathbf{L}_{1,n+1}\mathbf{S}_{n+2}^{-1} + \mathbf{L}_{-2,n+4}\mathbf{S}_{n+2}^{+2})^{-1}\mathbf{L}_{2,n}. \tag{57}$$

We perform an iteration procedure by starting from Eq. (52) for the case $n=0$,

$$0 = [\mathbf{L}_{-2,2}\mathbf{S}_0^{+2} + \mathbf{L}_{-1,1}\mathbf{S}_0^{+1} + (\mathbf{L}_{0,0} - \lambda\mathbf{I}) + \mathbf{L}_{1,-1}\mathbf{S}_0^{-1} + \mathbf{L}_{2,-2}\mathbf{S}_0^{-2}]\vec{\phi}_0^{\lambda}, \tag{58}$$

and by recursively inserting the recurrence relations of the ladder operators (54)–(57). Writing the successive inversions of the matrices formally as fractions, we may visualize this iteration procedure by a schematic representation of a pentadiagonal matrix valued continued fraction.

Thus far we have discussed the solution of the vector valued recurrence relation (52) in the pentadiagonal case for $m = \pm 1, \pm 2$. However, our method can be correspondingly extended to the general case where all Fourier components $\vec{\phi}_n^{\lambda}$ are coupled to each other. To this end we introduce ladder operators $\mathbf{S}_n^m$ with arbitrary m according to Eq. (50), where we identify $\mathbf{S}_n^0 = \mathbf{I}$. The homogeneous vector valued recurrence relation (46) then yields a corresponding one for the ladder operators $\mathbf{S}_n^m$:

$$0 = \sum_{k=-\infty}^{\infty} [\mathbf{L}_{k,n-k} - \delta_{k,0}(\lambda + in)\mathbf{I}]\mathbf{S}_n^{-k}. \tag{59}$$

An iteration procedure similar to Eqs. (54)–(57) finally leads to a homogeneous equation for the Fourier component $\vec{\phi}_0^{\lambda}$,

$$\mathbf{M}(\lambda)\vec{\phi}_0^{\lambda} = 0, \tag{60}$$

where the resulting matrix $\mathbf{M}(\lambda)$ consists of an infinite number of matrix valued continued fractions transcendentally depending on the Floquet eigenvalues λ. Therefore the Floquet eigenvalues λ are determined from the condition that the determinant of the matrix $\mathbf{M}(\lambda)$ vanish:

$$\det \mathbf{M}(\lambda) = 0. \tag{61}$$

Once the Floquet eigenvalues λ are known, the yet unknown Fourier component $\vec{\phi}_0^{\lambda}$ is determined up to a constant from solving the homogeneous equation (60). All Fourier components $\vec{\phi}_n^{\lambda}$ of the Floquet eigensolutions are then calculated by successively applying the ladder operators $\mathbf{S}_n^m$ starting with $\vec{\phi}_0^{\lambda}$:

$$\vec{\phi}_n^{\lambda} = \mathbf{S}_0^n \vec{\phi}_0^{\lambda}. \tag{62}$$

Note that the remaining normalization constant in $\vec{\phi}_0^{\lambda}$ has to be fixed by an adequate biorthonormality condition, which will be discussed in Sec. V D.

In applications, however, it is impossible to exactly evaluate the infinite number of matrix valued continued fractions. From an analytical point of view we can therefore expect that this solution method will allow us at most to approximately determine Floquet eigenvalues and the corresponding eigensolutions. To this end we recall that the starting point of our linear stability analysis, i.e., the time periodic reference state, is only known as a finite power series in the smallness parameter μ. As a consequence the whole calculation can be simplified by approximately neglecting higher order terms in the smallness parameter μ. In particular, it becomes sufficient to restrict the vector valued recurrence relation (46) to a finite number of terms, so that the subsequent iteration procedure only leads to a finite number of matrix valued continued fractions. Furthermore, each continued fraction can be evaluated in the leading order of the smallness parameter μ. In spite of these successive expansions, the continued fractions have the property that the approximate results rapidly converge towards the exact values if the leading order in the smallness parameter μ is increased.

B. Sketch of Risken's tridiagonal formulation

Following Risken [25] we show that the n diagonal matrix valued continued fractions can always be cast into a tridiagonal form. For the sake of simplicity we demonstrate this only for the pentadiagonal recurrence relation (49), but the general case is treated along similar lines. We start by distinguishing between even and odd indices n in the pentadiagonal recurrence relation (49):

$$0=\mathbf{L}_{-2,2n+2}\vec{\phi}^{\lambda}_{2n+2}+\mathbf{L}_{-1,2n+1}\vec{\phi}^{\lambda}_{2n+1}+[\mathbf{L}_{0,2n}-(\lambda+i2n)\mathbf{I}]\vec{\phi}^{\lambda}_{2n}+\mathbf{L}_{1,2n-1}\vec{\phi}^{\lambda}_{2n-1}+\mathbf{L}_{2,2n-2}\vec{\phi}^{\lambda}_{2n-2}, \tag{63}$$

$$0=\mathbf{L}_{-2,2n+3}\vec{\phi}^{\lambda}_{2n+3}+\mathbf{L}_{-1,2n+2}\vec{\phi}^{\lambda}_{2n+2}+\{\mathbf{L}_{0,2n+1}-[\lambda+i(2n+1)]\mathbf{I}\}\vec{\phi}^{\lambda}_{2n+1}+\mathbf{L}_{1,2n}\vec{\phi}^{\lambda}_{2n}+\mathbf{L}_{2,2n-1}\vec{\phi}^{\lambda}_{2n-1}. \tag{64}$$

We now construct new vectors according to the prescription

$$\vec{\Phi}^{\lambda}_{n+1}=\begin{pmatrix}\vec{\phi}^{\lambda}_{2n+2}\\ \vec{\phi}^{\lambda}_{2n+3}\end{pmatrix},\quad \vec{\Phi}^{\lambda}_{n}=\begin{pmatrix}\vec{\phi}^{\lambda}_{2n}\\ \vec{\phi}^{\lambda}_{2n+1}\end{pmatrix},\quad \vec{\Phi}^{\lambda}_{n-1}=\begin{pmatrix}\vec{\phi}^{\lambda}_{2n-2}\\ \vec{\phi}^{\lambda}_{2n-1}\end{pmatrix}. \tag{65}$$

Additionally we define the matrices

$$\mathbf{Q}_{-1,n+1}=\begin{pmatrix}\mathbf{L}_{-2,2n+2} & 0\\ \mathbf{L}_{-1,2n+2} & \mathbf{L}_{-2,2n+3}\end{pmatrix},$$

$$\mathbf{Q}_{1,n-1}=\begin{pmatrix}\mathbf{L}_{2,2n-2} & \mathbf{L}_{1,2n-1}\\ 0 & \mathbf{L}_{2,2n-1}\end{pmatrix},$$

$$\mathbf{Q}_{0,n}=\begin{pmatrix}[\mathbf{L}_{0,2n}-(\lambda+2in)\mathbf{I}] & \mathbf{L}_{-1,2n+1}\\ \mathbf{L}_{1,2n} & \{\mathbf{L}_{0,2n+1}-[\lambda+i(2n+1)]\mathbf{I}\}\end{pmatrix}. \tag{66}$$

Due to these definitions both pentadiagonal recurrence relations (63), (64) can be combined in the following way:

$$\mathbf{Q}_{-1,n+1}\vec{\Phi}^{\lambda}_{n+1}+\mathbf{Q}_{0,n}\vec{\Phi}^{\lambda}_{n}+\mathbf{Q}_{+1,n-1}\vec{\Phi}^{\lambda}_{n-1}=0. \tag{67}$$

Thus we have reached our goal of finding a tridiagonal vector valued recurrence relation. At this stage we define again ladder operators $\mathbf{R}_n^{\pm}$ with the property

$$\vec{\Phi}^{\lambda}_{n+1}=\mathbf{R}_n^{+}\vec{\Phi}^{\lambda}_{n},\quad \vec{\Phi}^{\lambda}_{n-1}=\mathbf{R}_n^{-}\vec{\Phi}^{\lambda}_{n}. \tag{68}$$

These ladder operators can be determined when we rewrite the tridiagonal recurrence relation (67) as

$$0=[\mathbf{Q}_{-1,n+1}\mathbf{R}_n^{+}+\mathbf{Q}_{0,n}]\vec{\Phi}^{\lambda}_{n}+\mathbf{Q}_{1,n-1}\vec{\Phi}^{\lambda}_{n-1} \tag{69}$$

and, similarly, as

$$0=\mathbf{Q}_{-1,n+1}\vec{\Phi}^{\lambda}_{n+1}+[\mathbf{Q}_{0,n}+\mathbf{Q}_{1,n-1}\mathbf{R}_n^{-}]\vec{\Phi}^{\lambda}_{n}. \tag{70}$$

Comparing these results with the original definitions (68) and shifting the index, we find relations for the ladder operators $\mathbf{R}_n^{\pm}$ themselves:

$$\mathbf{R}_n^{\mp}=-[\mathbf{Q}_{\pm1,n\mp2}\mathbf{R}^{\mp}_{n\mp1}+\mathbf{Q}_{0,n\mp1}]^{-1}\mathbf{Q}_{\mp1,n}. \tag{71}$$

Again it is sufficient for our purpose to solve the tridiagonal recurrence relation (67) for the case $n=0$:

$$0=[\mathbf{Q}_{-1,1}\mathbf{R}_0^{+}+\mathbf{Q}_{0,0}+\mathbf{Q}_{1,-1}\mathbf{R}_0^{-}]\vec{\Phi}^{\lambda}_{0}. \tag{72}$$

Repeated application of the operator relations (71) then yields a tridiagonal matrix valued continued fraction.

V. FORMULATION OF THE ADJOINT PROBLEM

In general, the linear infinitesimal generator $\mathcal{A}_L$ is not self-adjoint in the extended state space $\mathcal{C}$. Therefore it becomes necessary to define another extended state space $\mathcal{C}^{\dagger}$ dual to $\mathcal{C}$ and to investigate the properties of the adjoint infinitesimal generator $\mathcal{A}_L^{\dagger}$. In order to relate the linearized problem with its adjoint, it turns out that the canonical bilinear form for ordinary differential equations is not appropriate. In the case of delay differential equations a modified bilinear form has to be introduced.

A. The bilinear form

The choice of the canonical bilinear form for delay differential equations is motivated by the Fredholm alternative. To this end we consider the inhomogeneous version of Eq. (37),

$$\left(\left[\mathcal{A}_L-\frac{d}{d\xi}\right]\vec{q}_{\xi}\right)(\theta)=\vec{\chi}_{\xi}(\theta),\quad -\omega\tau\leqslant\theta\leqslant 0, \tag{73}$$

with a 2π periodic vector valued function $\vec{\chi}_{\xi}(\theta)=\vec{\chi}_{\xi+2\pi}(\theta)$. We try to construct a particular solution $\vec{q}_{\xi}(\theta)$ of Eq. (73) by the Floquet ansatz (41) with $\vec{\phi}^{\lambda}_{\xi}(\theta)=\vec{\phi}^{\lambda}_{\xi+2\pi}(\theta)$. Inserting the Fourier expansion (42) for $\vec{\phi}^{\lambda}_{\xi}(\theta)$ and a corresponding one for the inhomogeneity $\vec{\chi}_{\xi}(\theta)$, we obtain

$$\sum_{n=-\infty}^{\infty}([\mathcal{A}_L-\lambda-in]\vec{\phi}^{\lambda}_{n})(\theta)e^{in\xi}=\sum_{n=-\infty}^{\infty}\vec{\chi}_n(\theta)e^{in\xi}. \tag{74}$$

Taking into account the definition (38) of the infinitesimal generator $\mathcal{A}_L$ in the interval $-\omega\tau\leqslant\theta<0$, we conclude from Eq. (74) the general form of the Fourier component $\vec{\phi}^{\lambda}_{n}(\theta)$:

$$\vec{\phi}^{\lambda}_{n}(\theta)=\vec{\phi}^{\lambda}_{n}(0)e^{(\lambda+in)\theta}+\int_0^{\theta}ds\, e^{(\lambda+in)(\theta-s)}\vec{\chi}_n(s). \tag{75}$$

Correspondingly Eq. (74) determines for the point $\theta=0$ the yet unknown initial condition $\vec{\phi}^{\lambda}_{n}(0)$. With the definition (45) it fulfills an inhomogeneous vector valued recurrence relation:

$$\sum_{k=-\infty}^{\infty}[\mathbf{L}_{k,n-k}-\delta_{k,0}(\lambda+in)\mathbf{I}]\vec{\phi}^{\lambda}_{n-k}(0)=\vec{\chi}_n(0)-\sum_{k=-\infty}^{\infty}\int_{-\omega\tau}^{0}d\theta\int_0^{\theta}ds\, e^{(\lambda+i(n-k))(\theta-s)}\times\mathbf{\Omega}_k(\theta)\vec{\chi}_{n-k}(s). \tag{76}$$

This result suggests how to introduce both the dual space $\mathcal{C}^{\dagger}$ and the bilinear form. We assume that $\mathcal{C}^{\dagger}$ consists of n dimensional vector valued functions defined on the interval $[0,\omega\tau]$ and that the bilinear form is given by

$$[\vec{\psi}_\xi^\dagger(s),\vec{\phi}_\xi(\theta)]_\xi = \langle \vec{\psi}_\xi^\dagger(0),\vec{\phi}_\xi(0)\rangle - \int_{-\omega\tau}^0 d\theta \int_0^\theta ds \times \langle \vec{\psi}_\xi^\dagger(s-\theta),\mathbf{\Omega}_{\xi+s-\theta}(\theta)\vec{\phi}_\xi(s)\rangle \tag{77}$$

for all $\vec{\phi}_\xi \in \mathcal{C}$ and $\vec{\psi}_\xi^\dagger \in \mathcal{C}^\dagger$, where $\langle\rangle$ denotes the usual canonical scalar product. Note that each delay system and each time periodic reference state induces its own bilinear form due to Eq. (39). Furthermore, we observe that the explicit time dependent bilinear form (77) for a time periodic reference state reduces to the corresponding one for a time independent reference state [3].

With the bilinear form (77) the inhomogeneous recurrence relation (76) can be rewritten according to

$$\sum_{k=-\infty}^{\infty} [\mathbf{L}_{k,n-k} - \delta_{k,0}(\lambda + in)\mathbf{I}]\vec{\phi}_{n-k}^\lambda(0) = \frac{1}{2\pi}\int_0^{2\pi} d\xi (\mathbf{A}_{\xi,n}^\dagger(s),\vec{\chi}_\xi(\theta))_\xi , \tag{78}$$

where the matrix valued functions $\mathbf{A}_{\xi,n}^\dagger(s)$ are given by

$$\mathbf{A}_{\xi,n}^\dagger(s) = e^{-in\xi}e^{-(\lambda+in)s}\mathbf{I}, \quad 0 \leq s \leq \omega\tau. \tag{79}$$

Thus we obtain the following Fredholm alternative for solving the inhomogeneous equation (73). If the parameter λ does not coincide with a Floquet eigenvalue, we read off from Eq. (78) that there exists a unique solution. Otherwise we can only expect a solution if the inhomogeneity $\vec{\chi}_\xi$ fulfills a solvability condition that involves the bilinear form (77). This solvability condition will be concretized below after having defined the adjoint operator $\mathcal{A}_L^\dagger$ and its corresponding Floquet eigensolutions.

B. The adjoint operator

The bilinear form (77) can be applied to describe the evolution of the linearized delay system also in the dual extended state space $\mathcal{C}^\dagger$. To this end we require that the bilinear form between the state vector $\vec{q}_\xi \in \mathcal{C}$ and its dual $\vec{q}_\xi^\dagger \in \mathcal{C}^\dagger$ become time independent:

$$0 = \frac{d}{d\xi}(\vec{q}_\xi^\dagger(s),\vec{q}_\xi(\theta))_\xi . \tag{80}$$

As the bilinear form (77) does explicitly depend on the time ξ via the matrix valued density $\mathbf{\Omega}_\xi(\theta)$, we derive from Eqs. (37) and (80) the evolution equation in $\mathcal{C}^\dagger$,

$$\frac{d}{d\xi}\vec{q}_\xi^\dagger(s) = -(\mathcal{A}_L^\dagger \vec{q}_\xi^\dagger)(s), \quad 0 \leq s \leq \omega\tau, \tag{81}$$

where the adjoint infinitesimal generator $\mathcal{A}_L^\dagger$ obeys

$$(\mathcal{A}_L^\dagger \vec{q}_\xi^\dagger,\vec{q}_\xi)_\xi = (\vec{q}_\xi^\dagger,\mathcal{A}_L\vec{q}_\xi)_\xi - \int_{-\omega\tau}^0 d\theta\int_0^\theta ds \times \left\langle \vec{q}_\xi^\dagger(s-\theta), \frac{\partial}{\partial\xi}\mathbf{\Omega}_{\xi+s-\theta}(\theta)\vec{q}_\xi(s)\right\rangle . \tag{82}$$

When we use the definition (38) of the infinitesimal operator $\mathcal{A}_L$, we obtain from Eq. (82) after a partial integration the following expression for the adjoint infinitesimal generator $\mathcal{A}_L^\dagger$:

$$(\mathcal{A}_L^\dagger\vec{q}_\xi^\dagger)(s) = \begin{cases} -\dfrac{d}{ds}\vec{q}_\xi^\dagger(s), & 0 < s \leq \omega\tau, \\ \displaystyle\int_0^{\omega\tau} ds'\,\vec{q}_\xi^\dagger(s')\mathbf{\Omega}_{\xi+s'}(-s'), & s=0. \end{cases} \tag{83}$$

C. The adjoint recurrence relation

We are now in a position to solve the adjoint problem defined by Eqs. (81) and (83). In close analogy to the procedure in Sec. III C we perform the Floquet ansatz

$$\vec{q}_\xi^{\,\dagger}(s) = e^{-\lambda\xi}\vec{\psi}_\xi^{\dagger\lambda}(s), \tag{84}$$

with the 2π periodic adjoint Floquet eigensolution

$$\vec{\psi}_\xi^{\dagger\lambda}(s) = \sum_j \vec{\psi}_j^{\dagger\lambda}(s)e^{-ij\xi}. \tag{85}$$

Evaluating Eqs. (81) and (83) in the interval $0 < s \leq \omega\tau$ fixes the form of the Fourier components according to

$$\vec{\psi}_j^{\dagger\lambda}(s) = \vec{\psi}_j^{\dagger\lambda}e^{-(\lambda+ij)s}, \tag{86}$$

whereas the case $s=0$ leads to the corresponding homogeneous vector valued recurrence relation

$$0 = \sum_{k=-\infty}^{\infty} \vec{\psi}_{j+k}^{\dagger\lambda}[\mathbf{L}_{k,j} - \delta_{k,0}(\lambda+ij)\mathbf{I}]. \tag{87}$$

In order to solve Eq. (87) for the Fourier components $\vec{\psi}_j^{\dagger\lambda}$ of the adjoint Floquet eigensolutions and the respective Floquet eigenvalues λ, we proceed along lines similar to those in Sec. IV A. First we define adjoint ladder operators $\mathbf{Z}_j^m$ with arbitrary m and the identity $\mathbf{Z}_j^0 = \mathbf{I}$ in analogy to Eq. (50):

$$\vec{\psi}_{j+m}^{\dagger\lambda} = \vec{\psi}_j^{\dagger\lambda}\mathbf{Z}_j^m . \tag{88}$$

Inserting Eq. (88) in the homogeneous vector valued recurrence relation (87), we then obtain a corresponding one for the adjoint ladder operators:

$$0 = \sum_{k=-\infty}^{\infty} \mathbf{Z}_j^k[\mathbf{L}_{k,j} - \delta_{k,0}(\lambda+ij)\mathbf{I}]. \tag{89}$$

A careful comparison between Eqs. (59) and (89) reveals that the recurrence relations for the ladder operators $\mathbf{S}_n^m$ and their adjoint $\mathbf{Z}_j^m$ are not independent of each other. Indeed, they are mapped onto each other by the prescription

$$\mathbf{L}_{k,n-k}\mathbf{S}_n^{-k}=\mathbf{Z}_n^{-k}\mathbf{L}_{-k,n}. \tag{90}$$

This means that the adjoint ladder operators $\mathbf{Z}_j^m$ can immediately be calculated, once the ladder operators $\mathbf{S}_n^m$ are known. However, this does not imply that the solution of the adjoint problem directly follows from the linear problem. Iteratively inserting the operator recurrence relation (89) in the vector valued recurrence relation (87) for $j=0$ yields with Eq. (90)

$$\vec{\psi}_0^{\dagger\lambda}\mathbf{M}(\lambda)=0. \tag{91}$$

Thus the adjoint problem leads to the same condition (61) for the Floquet eigenvalues λ, but the Fourier component $\vec{\psi}_0^{\dagger\lambda}$ has to be determined independently from the Fourier component $\vec{\phi}_0^{\lambda}$ defined by Eq. (60).

With these definitions we are now able to concretize the Fredholm condition for solving the inhomogeneous equation (73). Multiplying Eq. (78) from the left with $\vec{\psi}_n^{\dagger\lambda}$, performing the summation over all n, and taking into account the homogeneous vector valued recurrence relation (87), we yield

$$\frac{1}{2\pi}\int_0^{2\pi}d\xi\left(\sum_{n=-\infty}^{\infty}\vec{\psi}_n^{\dagger\lambda}\mathbf{A}_{\xi,n}(s),\vec{\chi}_\xi(\theta)\right)_\xi=0. \tag{92}$$

Due to Eqs. (79), (85), and (86) this solvability condition takes the concise form

$$\frac{1}{2\pi}\int_0^{2\pi}d\xi(\vec{\psi}_\xi^{\dagger\lambda},\vec{\chi}_\xi(\theta))_\xi=0. \tag{93}$$

Only in the special case in which all quantities do not explicitly depend on the time ξ does this reduce to the usual Fredholm condition; i.e., the inhomogeneity $\vec{\chi}_\xi$ must be orthogonal to the respective eigensolution of the adjoint operator $\mathcal{A}_L^\dagger$ [3].

D. Biorthonormality relations

Finally we show that the Floquet eigensolutions $\vec{\phi}_\xi^\lambda$ and $\vec{\psi}_\xi^{\dagger\lambda}$ of the infinitesimal generator $\mathcal{A}_L$ and its dual $\mathcal{A}_L^\dagger$, respectively, can be chosen to form a biorthonormal set in $\mathcal{C}$ and $\mathcal{C}^\dagger$. First we derive the biorthogonality condition for different Floquet eigenvalues $\mu\neq\lambda$. Using the explicit expression for the bilinear form (77) and applying our previous results (40)–(42) and Eqs. (85) and (86), we obtain

$$\begin{aligned}(\vec{\psi}_\xi^{\dagger\lambda}(s),\vec{\phi}_\xi^{\mu}(\theta))_\xi=\sum_{n,j=-\infty}^{\infty}\Bigg\{&\langle\vec{\psi}_j^{\dagger\lambda},\vec{\phi}_n^{\mu}\rangle e^{i(n-j)\xi}\\ &-\sum_{k=-\infty}^{\infty}e^{i(n+k-j)\xi}\\ &\times\int_{-\omega\tau}^{0}d\theta\int_0^\theta ds\langle\vec{\psi}_j^{\dagger\lambda},\mathbf{\Omega}_k(\theta)\vec{\phi}_n^{\mu}\rangle\\ &\times e^{[\lambda+i(j-k)]\theta}e^{[\mu-\lambda+i(n+k-j)]s}\Bigg\}.\end{aligned} \tag{94}$$

An integration with respect to s yields for the second term on the right hand side

$$-\sum_{n,j,k=-\infty}^{\infty}e^{i(n+k-j)\xi}\int_{-\omega\tau}^{0}d\theta\langle\vec{\psi}_j^{\dagger\lambda},\mathbf{\Omega}_k(\theta)\vec{\phi}_n^{\mu}\rangle\times\frac{e^{(\mu+in)\theta}-e^{[\lambda+i(j-k)]\theta}}{\mu-\lambda+i(n+k-j)}. \tag{95}$$

With the definition (45) of the matrices $\mathbf{L}_{k,n}$ this reduces to

$$-\sum_{n,j,k=-\infty}^{\infty}e^{i(n+k-j)\xi}\frac{\langle\vec{\psi}_j^{\dagger\lambda},[\mathbf{L}_{k,n}-\mathbf{L}_{k,j-k}]\vec{\phi}_n^{\mu}\rangle}{\mu-\lambda+i(n+k-j)}, \tag{96}$$

so that we obtain from the homogeneous vector valued recurrence relations (46), (87),

$$-\sum_{n,j=-\infty}^{\infty}\langle\vec{\psi}_j^{\dagger\lambda},\vec{\phi}_n^{\mu}\rangle e^{i(n-j)\xi}. \tag{97}$$

From Eqs. (94)–(97) we conclude the biorthogonality for $\mu\neq\lambda$:

$$(\vec{\psi}_\xi^{\dagger\lambda}(s),\vec{\phi}_\xi^{\mu}(\theta))_\xi=0. \tag{98}$$

To normalize the biorthogonal set of Floquet eigenfunctions, we introduce a proper normalization constant in a symmetric way:

$$\vec{\phi}_n^{\lambda}=N_\lambda\vec{\Phi}_n^{\lambda},\quad \vec{\psi}_n^{\dagger\lambda}=N_\lambda\vec{\Psi}_n^{\dagger\lambda}. \tag{99}$$

From the requirement

$$(\vec{\psi}_\xi^{\dagger\lambda}(s),\vec{\phi}_\xi^{\lambda}(\theta))_\xi=1, \tag{100}$$

we then determine the normalization constant N_λ by performing similar calculations as above:

$$N_\lambda=\left[\sum_{n,j=-\infty}^{\infty}\left\langle\vec{\Psi}_j^{\dagger\lambda},\left(\delta_{j,n}-\int_{-\omega\tau}^{0}d\theta\,\theta e^{(\lambda+in)\theta}\mathbf{\Omega}_{j-n}(\theta)\right)\times\vec{\Phi}_n^{\lambda}\right\rangle\right]^{-1/2}. \tag{101}$$

As expected the normalization constant N_λ does not explicitly depend on the time ξ. Summarizing the results (98) and (100), the biorthonormality relation reads

$$(\vec{\psi}_{\xi}^{\dagger\lambda}(s),\vec{\phi}_{\xi}^{\mu}(\theta))_{\xi}=\delta_{\mu,\lambda}. \tag{102}$$

VI. SUMMARY AND CONCLUSIONS

The present paper was devoted to systematically developing a Floquet theory for delay differential equations. At first we approximately determined a time periodic reference state by extending two standard methods for ordinary differential equations, namely, the Poincaré-Lindstedt and the Shohat expansions. Then we tested the stability of this reference state by constructing Floquet eigensolutions and their corresponding eigenvalues from matrix valued continued fractions. Finally the Floquet theory was completed by studying the adjoint problem. The applicability of our Floquet theory was demonstrated in [27]. In particular, our analytical treatment provides a means of understanding the mechanism of the continuous control of chaos by self-controlling feedback [28,29]. Previous investigations have indicated that it becomes crucial to decide whether an observed stabilized limit cycle corresponds to an unstable cycle of the system or is produced by the control mechanism itself [27,30].

As the Floquet theory represents a linear stability analysis for a time periodic reference state, there still remains the nonlinear problem of constructing the normal form for an emerging instability. We expect that this problem can be tackled in a way similar to [3], where synergetic methods [26,24] are extended to investigate delay differential equations in the local neighborhood of a time independent reference state. Also close to the instability of a time periodic reference state the inherent time scale hierarchy should allow us to adiabatically eliminate the fast modes by using projectors that are induced by the bilinear form (77) of the linear stability analysis. As in [3] the resulting order parameter equations for the slow modes should turn out to be of the form of ordinary differential equations. We stress that the normal form theory is indispensable for classifying the instabilities of time periodic reference states. Whereas the linear stability analysis is sufficient to identify the instabilities of time independent reference states, this is no longer true for time periodic ones [24].

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